

# A note on pricing of contingent claims under $G$ -expectation

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March 20, 2013

**Abstract.** In this paper, we study the pricing of contingent claims under  $G$ -expectation. In order to accomodate volatility uncertainty, the price of the risky security is supposed to be governed by a general linear stochastic differential equation (SDE) driven by  $G$ -Brownian motion. Utilizing the recently developed results of Backward SDE driven by  $G$ -Brownian motion, we obtain the superhedging and subhedging prices of a given contingent claim. Explicit results in the Markovian case are also derived.

**Key words:** Pricing of contingent claims, volatility uncertainty,  $G$ -Brownian motion, Backward SDEs

**MSC-classification:** 60H30, 91G20

## 1 Introduction

It is well known that the Black-Scholes formula depends on the underlying volatility. Since it is difficult to forecast the prospective volatility process in practice, it is natural to permit volatility uncertainty in contingent claim pricing models (see [1]).

Motivated by measuring risk and other financial problems of volatility uncertainty, Peng [19] introduced the notion of sublinear expectation space, which is a generalization of probability space. As a typical case, Peng studied a fully nonlinear expectation, called  $G$ -expectation  $\mathbb{E}[\cdot]$  (see [23] and the references therein), and the corresponding time-conditional expectation  $\hat{\mathbb{E}}_t[\cdot]$  on a space of random variables completed under the norm  $\hat{\mathbb{E}}[\|\cdot\|^p]^{1/p}$ . Under this  $G$ -expectation framework ( $G$ -framework for short) a new type of Brownian motion called  $G$ -Brownian motion was constructed. The stochastic calculus with respect to the  $G$ -Brownian motion has been established. For a recent account and development of  $G$ -expectation theory and its applications we refer the reader to [5, 6, 13, 17, 18, 24, 25, 26, 28, 29].

There are other recent advances and their applications in stochastic calculus which consists of mutually singular probability measures. For instance, Denis

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and Martini [3] developed quasi-sure stochastic analysis and Soner et al. [27] have obtained a deep result of existence and uniqueness theorem of 2BSDE. Various stochastic control (game) problems and the applications in finance are studied in [10, 11, 12, 14, 15].

In this paper, we suppose that there are a riskless asset a risky security in a financial market. Different from the existing literatures (see [2, 6, 30, 31]), the price  $S_t$  to the risky security is governed by

$$dS_t = \eta_t S_t dt + \mu_t S_t d\langle B \rangle_t + \sigma_t S_t dB_t,$$

where  $B$  is a  $G$ -Brownian motion. For a given contingent claim  $\xi \in L_G^2(\Omega)$  with maturity time  $T$ , we obtain its superhedging and subhedging prices. Explicit results in the Markovian case are also derived. Our study bases on the recently developed BSDE driven by  $G$ -Brownian motion in [7] and [8]:

$$\begin{aligned} Y_t = \xi &+ \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s, Z_s) d\langle B \rangle_s \\ &- \int_t^T Z_s dB_s - (K_T - K_t). \end{aligned}$$

We mainly utilize the existence and uniqueness theorem in [7] and some important properties such as comparison theorem, Feynman-Kac formula and Girsanov transformation in [8].

The paper is organized as follows. In section 2, we formulate our contingent claim pricing problem. The main results are given in section 3. In the Appendix, we present some fundamental results on  $G$ -expectation theory and give proofs of the comparison theorem of SDE driven by  $G$ -Brownian motion and the Girsanov transformation in our context.

## 2 Statement of the problem

There are a riskless asset with return  $r_t$  and a risky security in a financial market. The price  $S_t$  to the risky securities is given by

$$dS_t = \eta_t S_t dt + \mu_t S_t d\langle B \rangle_t + \sigma_t S_t dB_t, \quad t \leq T, \quad (2.1)$$

where  $(\eta_t)$ ,  $(\mu_t)$ ,  $(\sigma_t)$  and  $(\sigma_t^{-1})$  are all bounded processes in  $M_G^2(0, T)$ . The readers may refer to the Appendix to find the basic definitions and fundamental results in the  $G$ -framework.

We denote the wealth process by  $(Y_t)$  and the amount of money invested in the security by  $(\psi_t)$  at time  $t$ . Then the wealth process follows

$$dY_t = r_t Y_t dt + \psi_t [(\eta_t - r_t) dt + \mu_t d\langle B \rangle_t] + \psi_t \sigma_t dB_t. \quad (2.2)$$

Set  $Z_t = \psi_t \sigma_t$ ,  $b_t = \sigma_t^{-1}(\eta_t - r_t)$  and  $d_t = \sigma_t^{-1} \mu_t$ . Then (2.2) becomes

$$dY_t = r_t Y_t dt + b_t Z_t dt + d_t Z_t d\langle B \rangle_t + Z_t dB_t, \quad (2.3)$$

here  $Z$  is called the portfolio. In this note, we suppose that every  $Z \in M_G^2(0, T)$  is an admissible portfolio.

At the initial time  $\tau \in [0, T]$ , consider an investor with initial wealth  $\eta \in L_G^2(\Omega_\tau)$  and denote by  $Y^{\eta, Z, \tau}$  the unique solution of the following SDE:

$$\begin{cases} dY_t^{\eta, Z, \tau} = r_t Y_t^{\eta, Z, \tau} dt + b_t Z_t dt + d_t Z_t d\langle B \rangle_t + Z_t dB_t, & t \in [\tau, T], \\ Y_\tau^{\eta, Z, \tau} = \eta, \end{cases} \quad (2.4)$$

where  $Z \in M_G^2(\tau, T)$  is a given portfolio. For a contingent claim  $\xi \in L_G^\beta(\Omega_T)$  with  $\beta > 2$ , we define the superhedging set

$$\mathcal{U}_\tau = \{\eta \in L_G^2(\Omega_\tau) : \exists Z \in M_G^2(\tau, T) \text{ such that } Y_T^{\eta, Z, \tau} \geq \xi, \text{ q.s.}\}$$

and the *superhedging price*  $\overline{S}_\tau = \text{ess inf}\{\eta : \eta \in \mathcal{U}_\tau\}$ . Similarly define the subhedging set

$$\mathcal{L}_\tau = \{\eta \in L_G^2(\Omega_\tau) : \exists Z \in M_G^2(0, T) \text{ such that } Y_T^{-\eta, Z, \tau} \geq -\xi, \text{ q.s.}\}$$

and the *subhedging price*  $\underline{S}_\tau = \text{ess sup}\{\eta : \eta \in \mathcal{L}_\tau\}$ .

**Remark 2.1** For  $\tau = 0$ ,  $\mathcal{U}_0 \subset \mathbb{R}$ , thus  $\overline{S}_0 = \inf\{y \in \mathbb{R} : y \in \mathcal{U}_0\}$  is well defined. For  $\tau > 0$ ,  $\overline{S}_\tau = \text{ess inf}\{\eta : \eta \in \mathcal{U}_\tau\}$  is defined in the following sense:

- (1)  $\overline{S}_\tau \in L_G^2(\Omega_\tau)$ ;
- (2) For each  $\eta \in \mathcal{U}_\tau$ , we have  $\eta \geq \overline{S}_\tau$  q.s.;
- (3) If  $\zeta \in L_G^2(\Omega_\tau)$  such that  $\zeta \leq \eta$  q.s. for each  $\eta \in \mathcal{U}_\tau$ , then  $\overline{S}_\tau \geq \zeta$  q.s..

In this note, we will show that  $\overline{S}_\tau$  is well-posed which is non-trivial due to the non-dominated probability measures in  $\mathcal{P}$ . Similarly,  $\underline{S}_\tau$  is well defined.

## 3 Main results

### 3.1 State price process

We consider the following  $G$ -BSDE:

$$Y_t = \xi - \int_t^T (r_s Y_s + b_s Z_s) ds - \int_t^T d_s Z_s d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t), \quad t \leq T. \quad (3.1)$$

In order to introduce the state price process which can be used to solve the  $G$ -BSDE (3.1), we construct an auxiliary extended  $\tilde{G}$ -expectation space  $(\tilde{\Omega}_T, L_G^2(\tilde{\Omega}_T), \hat{\mathbb{E}}^{\tilde{G}})$  with  $\tilde{\Omega}_T = C_0([0, T], \mathbb{R}^2)$  and

$$\tilde{G}(A) = \frac{1}{2} \sup_{\sigma^2 \leq v \leq \bar{\sigma}^2} \text{tr} \left[ A \begin{bmatrix} v & 1 \\ 1 & v^{-1} \end{bmatrix} \right], \quad A \in \mathbb{S}_2.$$

Let  $\{(B_t, \tilde{B}_t)\}$  be the canonical process in the extended space (see [8]). Note that  $\langle B, \tilde{B} \rangle_t = t$ .

By the state price process we mean the unique solution  $\pi = (\pi_t)$  to

$$d\pi_t/\pi_t = -r_t dt - b_t d\tilde{B}_t - d_t dB_t, \quad \pi_0 = 1, \quad (3.2)$$

which admits a closed form (see [8]): for  $0 \leq t \leq T$ ,

$$\begin{aligned} \pi_t = & \exp\left\{-\int_0^t (r_s + b_s d_s) ds\right\} \exp\left\{-\int_0^t b_s d\tilde{B}_s - \frac{1}{2} \int_0^t b_s^2 d\langle \tilde{B} \rangle_s\right\} \\ & \exp\left\{-\int_0^t d_s dB_s - \frac{1}{2} \int_0^t d_s^2 d\langle B \rangle_s\right\}. \end{aligned}$$

By applying Itô's formula to  $\pi_t Y_t$ , we obtain

$$Y_t = \hat{\mathbb{E}}_t^{\tilde{G}}\left[\frac{\pi_T}{\pi_t} \xi\right], \quad t \leq T. \quad (3.3)$$

### 3.2 Hedging prices

**Theorem 3.1 (Hedging prices)** *Let  $\xi \in L_G^2(\Omega_T)$  be a contingent claim. Suppose that  $(r_t)$ ,  $(\eta_t)$ ,  $(\mu_t)$ ,  $(\sigma_t)$  and  $(\sigma_t^{-1})$  are bounded processes in  $M_G^2(0, T)$ . Then the superhedging and subhedging prices at any time  $\tau$  are given by*

$$\overline{S}_\tau = \hat{\mathbb{E}}_\tau^{\tilde{G}}\left[\frac{\pi_T}{\pi_\tau} \xi\right]$$

and

$$\underline{S}_\tau = -\hat{\mathbb{E}}_\tau^{\tilde{G}}\left[-\frac{\pi_T}{\pi_\tau} \xi\right].$$

**Proof.** By the definition of subhedging price, it is easy to get  $\underline{S}_\tau$  from the superhedging price  $\overline{S}_\tau$ . Thus we only need to prove the superhedging price.

**Step 1:** We first show that for any  $\eta \in \mathcal{U}_\tau$ ,

$$\eta \geq \hat{\mathbb{E}}_\tau^{\tilde{G}}\left[\frac{\pi_T}{\pi_\tau} \xi\right], \quad \text{q.s..}$$

If  $\eta \in \mathcal{U}_\tau$ , then there exists a  $Z \in M_G^2(\tau, T)$  such that  $Y_T^{\eta, Z, \tau} \geq \xi$ . Thus

$$Y_t^{\eta, Z, \tau} = Y_T^{\eta, Z, \tau} - \int_t^T (r_s Y_s^{\eta, Z, \tau} + b_s Z_s) ds - \int_t^T d_s Z_s d\langle B \rangle_s - \int_t^T Z_s dB_s, \quad t \in [\tau, T]. \quad (3.4)$$

Let  $(\bar{Y}_t, \bar{Z}_t, \bar{K}_t)_{t \leq T}$  be the solution of the  $G$ -BSDE (3.1) corresponding to the terminal value  $Y_T^{\eta, Z, \tau}$ . Then by (3.4) we get  $(\bar{Y}_t, \bar{Z}_t, \bar{K}_t) = (Y_t^{\eta, Z, \tau}, Z_t, 0)$  for  $t \in [\tau, T]$ . Let  $(\tilde{Y}_t, \tilde{Z}_t, \tilde{K}_t)_{t \leq T}$  be the solution of the  $G$ -BSDE (3.1) corresponding to the terminal value  $\xi$ . Then by (3.3) we have  $\tilde{Y}_t = \hat{\mathbb{E}}_t^{\tilde{G}}[\frac{\pi_T}{\pi_t} \xi]$  for  $t \leq T$ . Note that  $Y_T^{\eta, Z, \tau} \geq \xi$ , then by the comparison theorem of  $G$ -BSDEs (see [8]) we obtain

$$\bar{Y}_\tau = Y_\tau^{\eta, Z, \tau} = \eta \geq \tilde{Y}_\tau = \hat{\mathbb{E}}_\tau^{\tilde{G}}\left[\frac{\pi_T}{\pi_\tau} \xi\right], \quad \text{q.s..}$$

**Step 2:** We now prove that  $\hat{\mathbb{E}}_\tau^{\tilde{G}}[\frac{\pi_T}{\pi_\tau} \xi] = \tilde{Y}_\tau \in \mathcal{U}_\tau$ .

For this purpose, we consider the following wealth process  $(\hat{Y}_t)_{t \in [\tau, T]}$  with the initial wealth  $\tilde{Y}_\tau$  and portfolio  $\tilde{Z}$ :

$$\hat{Y}_t = \tilde{Y}_\tau + \int_\tau^t (r_s \hat{Y}_s + b_s \tilde{Z}_s) ds + \int_\tau^t d_s \tilde{Z}_s d\langle B \rangle_s + \int_\tau^t \tilde{Z}_s dB_s, \quad t \in [\tau, T].$$

On the other hand,  $(\tilde{Y}_t, \tilde{Z}_t, \tilde{K}_t)_{t \leq T}$  is the solution of the  $G$ -BSDE (3.1) corresponding to the terminal value  $\xi$ . Thus we get

$$\tilde{Y}_t = \tilde{Y}_\tau + \int_\tau^t (r_s \tilde{Y}_s + b_s \tilde{Z}_s) ds + \int_\tau^t d_s \tilde{Z}_s d\langle B \rangle_s + \int_\tau^t \tilde{Z}_s dB_s + \tilde{K}_t - \tilde{K}_\tau, t \in [\tau, T].$$

Note that  $\tilde{K}$  is a decreasing process, then by the comparison theorem of SDE (see Appendix) we obtain  $\tilde{Y}_T \geq \tilde{Y}_\tau = \xi$  q.s., which implies that  $\tilde{Y}_\tau \in \mathcal{U}_\tau$ .

This completes the proof. ■

**Remark 3.2** In the special case where  $\xi$  can be perfectly hedged, that is, there exist  $y$  and  $Z$  such that  $Y_T^{y,Z,0} = \xi$ , then

$$\overline{S}_0 = \underline{S}_0 = \hat{\mathbb{E}}_t^{\tilde{G}}[\frac{\pi_T}{\pi_\tau} \xi] = -\hat{\mathbb{E}}_t^{\tilde{G}}[-\frac{\pi_T}{\pi_\tau} \xi].$$

**Remark 3.3** Vorbrink (2010) obtains a characterization of hedging prices under  $G$ -expectation. However, in place of our assumption, he adopts the strong assumption that  $\eta_t = r_t$  and  $\mu_t = 0$ , so  $\pi_t = \exp\{-\int_0^t r_s ds\}$ .

### 3.3 Some special cases

Suppose that  $(r_t)$ ,  $(\sigma_t)$  and  $(\sigma_t^{-1})$  are deterministic continuous functions on the time interval  $[0, T]$ .  $\xi = \Phi(S_T)$  is a contingent claim, where  $\Phi : \mathbb{R} \rightarrow \mathbb{R}$  is a local Lipschitz function, i.e., there exist a constant  $L > 0$  and an positive integer  $m$  such that

$$|\Phi(x) - \Phi(x')| \leq L(1 + |x|^m + |x'|^m)|x - x'|.$$

We consider the following  $G$ -BSDEs:

$$Y_t = \Phi(S_T) - \int_t^T (r_s Y_s + b_s Z_s) ds - \int_t^T d_s Z_s d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t), \quad (3.5)$$

$$\bar{Y}_t = -\Phi(S_T) - \int_t^T (r_s \bar{Y}_s + b_s \bar{Z}_s) ds - \int_t^T d_s \bar{Z}_s d\langle B \rangle_s - \int_t^T \bar{Z}_s dB_s - (\bar{K}_T - \bar{K}_t). \quad (3.6)$$

By (3.3) and Theorem 3.1, we have  $\overline{S}_\tau = Y_\tau$  and  $\underline{S}_\tau = -\bar{Y}_\tau$ .

By applying Itô's formula to  $\exp\{-\int_0^t r_s ds\} Y_t$ , we obtain that  $\tilde{Y}_t = \exp\{-\int_0^t r_s ds\} Y_t$ ,  $\tilde{Z}_t = \exp\{-\int_0^t r_s ds\} Z_t$  and  $\tilde{K}_t = \int_0^t \exp\{-\int_0^u r_s ds\} dK_u$  is the solution of the following  $G$ -BSDE:

$$\tilde{Y}_t = \exp\{-\int_0^T r_s ds\} \Phi(S_T) - \int_t^T b_s \tilde{Z}_s ds - \int_t^T d_s \tilde{Z}_s d\langle B \rangle_s - \int_t^T \tilde{Z}_s dB_s - (\tilde{K}_T - \tilde{K}_t). \quad (3.7)$$

By the Girsanov transformation (see Appendix), we can define a consistent sublinear expectation  $(\tilde{\mathbb{E}}_t[\cdot])_{t \leq T}$  such that  $\tilde{B}_t = B_t + \int_0^t b_s ds + \int_0^t d_s d\langle B \rangle_s$  is

a  $G$ -Brownian motion and  $\tilde{K}_t$  is a martingale under  $\tilde{\mathbb{E}}$ . Thus equation (3.7) becomes

$$\tilde{Y}_t + (\tilde{K}_T - \tilde{K}_t) = \exp\left\{-\int_0^T r_s ds\right\} \Phi(S_T) - \int_t^T \tilde{Z}_s d\tilde{B}_s. \quad (3.8)$$

Taking  $\tilde{\mathbb{E}}_t$  on both sides of equation (3.8), we obtain

$$\begin{aligned} \tilde{Y}_t &= \tilde{\mathbb{E}}_t\left[\exp\left\{-\int_0^T r_s ds\right\} \Phi(S_T)\right] \\ &= \exp\left\{-\int_0^T r_s ds\right\} \tilde{\mathbb{E}}_t[\Phi(S_T)] \\ &= \exp\left\{-\int_0^T r_s ds\right\} \tilde{\mathbb{E}}_t\left[\Phi\left(S_t \exp\left(\int_t^T r_s ds - \frac{1}{2} \int_t^T \sigma_s^2 d\langle B \rangle_s + \int_t^T \sigma_s d\tilde{B}_s\right)\right)\right] \\ &= \exp\left\{-\int_0^T r_s ds\right\} \tilde{\mathbb{E}}_t\left[\Phi\left(S_t \exp\left(\int_t^T r_s ds - \frac{1}{2} \int_t^T \sigma_s^2 d\langle \tilde{B} \rangle_s + \int_t^T \sigma_s d\tilde{B}_s\right)\right)\right] \\ &= \exp\left\{-\int_0^T r_s ds\right\} \tilde{\mathbb{E}}_t\left[\Phi\left(x \exp\left(\int_t^T r_s ds - \frac{1}{2} \int_t^T \sigma_s^2 d\langle \tilde{B} \rangle_s + \int_t^T \sigma_s d\tilde{B}_s\right)\right)\right]_{x=S_t} \\ &= \exp\left\{-\int_0^T r_s ds\right\} \tilde{\mathbb{E}}_t\left[\Phi\left(x \exp\left(\int_t^T r_s ds - \frac{1}{2} \int_t^T \sigma_s^2 d\langle \tilde{B} \rangle_s + \int_t^T \sigma_s d\tilde{B}_s\right)\right)\right]_{x=S_t} \\ &= \exp\left\{-\int_0^T r_s ds\right\} \hat{\mathbb{E}}_t\left[\Phi\left(x \exp\left(\int_t^T r_s ds - \frac{1}{2} \int_t^T \sigma_s^2 d\langle B \rangle_s + \int_t^T \sigma_s dB_s\right)\right)\right]_{x=S_t}. \end{aligned}$$

Thus we can get the following theorem.

**Theorem 3.4** *Suppose that  $(r_t)$ ,  $(\sigma_t)$  and  $(\sigma_t^{-1})$  are deterministic continuous functions on the time interval  $[0, T]$ . Let  $\xi = \Phi(S_T)$  be a contingent claim, where  $\Phi : \mathbb{R} \rightarrow \mathbb{R}$  is a local Lipschitz function. Then*

$$\overline{S}_\tau = \exp\left\{-\int_\tau^T r_s ds\right\} \hat{\mathbb{E}}\left[\Phi\left(x \exp\left(\int_\tau^T r_s ds - \frac{1}{2} \int_\tau^T \sigma_s^2 d\langle B \rangle_s + \int_\tau^T \sigma_s dB_s\right)\right)\right]_{x=S_t} \quad (3.9)$$

and

$$\underline{S}_\tau = -\exp\left\{-\int_\tau^T r_s ds\right\} \hat{\mathbb{E}}\left[-\Phi\left(x \exp\left(\int_\tau^T r_s ds - \frac{1}{2} \int_\tau^T \sigma_s^2 d\langle B \rangle_s + \int_\tau^T \sigma_s dB_s\right)\right)\right]_{x=S_t}. \quad (3.10)$$

For each  $(\tau, x) \in [0, T] \times \mathbb{R}$ , we set

$$u(\tau, x) = \exp\left\{-\int_\tau^T r_s ds\right\} \hat{\mathbb{E}}\left[\Phi\left(x \exp\left(\int_\tau^T r_s ds - \frac{1}{2} \int_\tau^T \sigma_s^2 d\langle B \rangle_s + \int_\tau^T \sigma_s dB_s\right)\right)\right]. \quad (3.11)$$

Then  $\overline{S}_\tau = u(\tau, S_t)$  and  $u$  is the unique viscosity solution of the following PDE (see Theorem 4.5 in [8]):

$$\begin{cases} \partial_t u + G((\sigma_t x)^2 \partial_{xx}^2 u) + r_t x \partial_x u - r_t u = 0, \\ u(T, x) = \Phi(x). \end{cases}$$

**Example 3.5** We study the super and subhedging prices of a European call option. Let the parameters in equation (2.1) be constants, i.e.

$$\eta_t := \eta, \mu_t := \mu \text{ and } \sigma_t := 1.$$

Then the price process  $(S_t)$  becomes

$$dS_t = \eta S_t dt + \mu S_t d\langle B \rangle_t + S_t dB_t.$$

Suppose further that  $r_t \equiv r$  and  $r$  is a constant. Thus  $b_t = \eta - r$ ,  $d_t = \mu$  and the state price is

$$\pi_t = \exp\{-\mu(\eta-r)t\} \exp\{-rt - (\eta-r)\tilde{B}_t - \frac{1}{2}(\eta-r)^2\langle \tilde{B} \rangle_t\} \exp\{-\mu B_t - \frac{1}{2}\mu^2\langle B \rangle_t\}.$$

Consider a European call option on the risky security that matures at date  $T$  and has exercise price  $K$ . The super and subhedging prices at  $t$  can be written in the form  $\bar{c}(S_t, t)$  and  $\underline{c}(S_t, t)$  respectively. At the maturity date,

$$\bar{c}(S_T, T) = \underline{c}(S_T, T) = \max[0, S_T - K] \equiv \Phi(S_T).$$

By Theorem 3.1,

$$\bar{c}(S_t, t) = \hat{\mathbb{E}}_t^{\tilde{G}}\left[\frac{\pi_T}{\pi_t}\Phi(S_T)\right]$$

and

$$\underline{c}(S_t, t) = -\hat{\mathbb{E}}_t^{\tilde{G}}\left[-\frac{\pi_T}{\pi_t}\Phi(S_T)\right].$$

By the PDE approach, we obtain the following equations:

$$\partial_t \bar{c} + \sup_{\underline{\sigma}^2 \leq v \leq \bar{\sigma}^2} \left\{ \frac{1}{2} v S^2 \partial_{SS}^2 \bar{c} \right\} + r S \partial_S \bar{c} - r \bar{c} = 0, \quad \bar{c}(S, T) = \Phi(S)$$

and

$$\partial_t \underline{c} - \sup_{\underline{\sigma}^2 \leq v \leq \bar{\sigma}^2} \left\{ -\frac{1}{2} v S^2 \partial_{SS}^2 \underline{c} \right\} + r S \partial_S \underline{c} - r \underline{c} = 0, \quad \underline{c}(S, T) = \Phi(S).$$

Because  $\Phi(\cdot)$  is convex, so is  $\bar{c}(\cdot, t)$ . It follows that the respective suprema in the above equations are achieved at  $\bar{\sigma}^2$  and  $\underline{\sigma}^2$ , and we obtain

$$\partial_t \bar{c} + \frac{1}{2} \bar{\sigma}^2 S^2 \partial_{SS}^2 \bar{c} + r S \partial_S \bar{c} - r \bar{c} = 0, \quad \bar{c}(S, T) = \Phi(S)$$

and

$$\partial_t \underline{c} + \frac{1}{2} \underline{\sigma}^2 S^2 \partial_{SS}^2 \underline{c} + r S \partial_S \underline{c} - r \underline{c} = 0, \quad \underline{c}(S, T) = \Phi(S).$$

Therefore,

$$\bar{c}(S_t, t) = E^{P^{\bar{\sigma}}}\left[\frac{\pi_T}{\pi_t}\Phi(S_T) \mid \mathcal{F}_t\right]$$

and

$$\underline{c}(S_t, t) = E^{P^{\underline{\sigma}}}\left[\frac{\pi_T}{\pi_t}\Phi(S_T) \mid \mathcal{F}_t\right].$$

In other words, the super and subhedging prices are the Black-Scholes prices with volatilities  $\bar{\sigma}$  and  $\underline{\sigma}$  respectively.

**Remark 3.6** In the above example, we find that the super and subhedging prices are independent of  $\eta$  and  $\mu$ .

## 4 Appendix

We review some basic notions and results of  $G$ -expectation, the related spaces of random variables and the backward stochastic differential equations driven by a  $G$ -Brownian motion. The readers may refer to [7], [19], [20], [21], [22], [23] for more details.

**Definition 4.1** Let  $\Omega$  be a given set and let  $\mathcal{H}$  be a vector lattice of real valued functions defined on  $\Omega$ , namely  $c \in \mathcal{H}$  for each constant  $c$  and  $|X| \in \mathcal{H}$  if  $X \in \mathcal{H}$ .  $\mathcal{H}$  is considered as the space of random variables. A sublinear expectation  $\hat{\mathbb{E}}$  on  $\mathcal{H}$  is a functional  $\hat{\mathbb{E}} : \mathcal{H} \rightarrow \mathbb{R}$  satisfying the following properties: for all  $X, Y \in \mathcal{H}$ , we have

- (a) *Monotonicity:* If  $X \geq Y$  then  $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$ ;
- (b) *Constant preservation:*  $\hat{\mathbb{E}}[c] = c$ ;
- (c) *Sub-additivity:*  $\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$ ;
- (d) *Positive homogeneity:*  $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$  for each  $\lambda \geq 0$ .

$(\Omega, \mathcal{H}, \hat{\mathbb{E}})$  is called a sublinear expectation space.

**Definition 4.2** Let  $X_1$  and  $X_2$  be two  $n$ -dimensional random vectors defined respectively in sublinear expectation spaces  $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$  and  $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$ . They are called identically distributed, denoted by  $X_1 \stackrel{d}{=} X_2$ , if  $\hat{\mathbb{E}}_1[\varphi(X_1)] = \hat{\mathbb{E}}_2[\varphi(X_2)]$ , for all  $\varphi \in C_{l.Lip}(\mathbb{R}^n)$ , where  $C_{l.Lip}(\mathbb{R}^n)$  is the space of real continuous functions defined on  $\mathbb{R}^n$  such that

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y| \quad \text{for all } x, y \in \mathbb{R}^n,$$

where  $k$  and  $C$  depend only on  $\varphi$ .

**Definition 4.3** In a sublinear expectation space  $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ , a random vector  $Y = (Y_1, \dots, Y_n)$ ,  $Y_i \in \mathcal{H}$ , is said to be independent of another random vector  $X = (X_1, \dots, X_m)$ ,  $X_i \in \mathcal{H}$  under  $\hat{\mathbb{E}}[\cdot]$ , denoted by  $Y \perp X$ , if for every test function  $\varphi \in C_{l.Lip}(\mathbb{R}^m \times \mathbb{R}^n)$  we have  $\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, Y)]_{x=X}]$ .

**Definition 4.4** ( $G$ -normal distribution) A  $d$ -dimensional random vector  $X = (X_1, \dots, X_d)$  in a sublinear expectation space  $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$  is called  $G$ -normally distributed if for each  $a, b \geq 0$  we have

$$aX + b\bar{X} \stackrel{d}{=} \sqrt{a^2 + b^2}X,$$

where  $\bar{X}$  is an independent copy of  $X$ , i.e.,  $\bar{X} \stackrel{d}{=} X$  and  $\bar{X} \perp X$ . Here the letter  $G$  denotes the function

$$G(A) := \frac{1}{2} \hat{\mathbb{E}}[AX, X] : \mathbb{S}_d \rightarrow \mathbb{R},$$

where  $\mathbb{S}_d$  denotes the collection of  $d \times d$  symmetric matrices.



Peng [22] showed that  $X = (X_1, \dots, X_d)$  is  $G$ -normally distributed if and only if for each  $\varphi \in C_{l.Lip}(\mathbb{R}^d)$ ,  $u(t, x) := \hat{\mathbb{E}}[\varphi(x + \sqrt{t}X)]$ ,  $(t, x) \in [0, \infty) \times \mathbb{R}^d$ , is the solution of the following  $G$ -heat equation:

$$\partial_t u - G(D_x^2 u) = 0, \quad u(0, x) = \varphi(x).$$

The function  $G(\cdot) : \mathbb{S}_d \rightarrow \mathbb{R}$  is a monotonic, sublinear mapping on  $\mathbb{S}_d$  and  $G(A) = \frac{1}{2}\hat{\mathbb{E}}[\langle AX, X \rangle] \leq \frac{1}{2}|A|\hat{\mathbb{E}}[|X|^2] =: \frac{1}{2}|A|\bar{\sigma}^2$  implies that there exists a bounded, convex and closed subset  $\Gamma \subset \mathbb{S}_d^+$  such that

$$G(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} \text{tr}[\gamma A],$$

where  $\mathbb{S}_d^+$  denotes the collection of nonnegative elements in  $\mathbb{S}_d$ .

In this paper, we only consider non-degenerate  $G$ -normal distribution, i.e., there exists some  $\underline{\sigma}^2 > 0$  such that  $G(A) - G(B) \geq \underline{\sigma}^2 \text{tr}[A - B]$  for any  $A \geq B$ .

**Definition 4.5** *i) Let  $\Omega_T = C_0([0, T]; \mathbb{R}^d)$ , the space of real valued continuous functions on  $[0, T]$  with  $\omega_0 = 0$ , be endowed with the supremum norm and let  $B_t(\omega) = \omega_t$  be the canonical process. Set*

$$\mathcal{H}_T^0 := \{\varphi(B_{t_1}, \dots, B_{t_n}) : n \geq 1, t_1, \dots, t_n \in [0, T], \varphi \in C_{l.Lip}(\mathbb{R}^{d \times n})\}.$$

*Let  $G : \mathbb{S}_d \rightarrow \mathbb{R}$  be a given monotonic and sublinear function.  $G$ -expectation is a sublinear expectation defined by*

$$\hat{\mathbb{E}}[X] = \tilde{\mathbb{E}}[\varphi(\sqrt{t_1 - t_0}\xi_1, \dots, \sqrt{t_m - t_{m-1}}\xi_m)],$$

*for all  $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_m} - B_{t_{m-1}})$ , where  $\xi_1, \dots, \xi_n$  are identically distributed  $d$ -dimensional  $G$ -normally distributed random vectors in a sublinear expectation space  $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{\mathbb{E}})$  such that  $\xi_{i+1}$  is independent of  $(\xi_1, \dots, \xi_i)$  for every  $i = 1, \dots, m-1$ . The corresponding canonical process  $B_t = (B_t^i)_{i=1}^d$  is called a  $G$ -Brownian motion.*

*ii) Let us define the conditional  $G$ -expectation  $\hat{\mathbb{E}}_t$  of  $\xi \in \mathcal{H}_T^0$  knowing  $\mathcal{H}_t^0$ , for  $t \in [0, T]$ . Without loss of generality we can assume that  $\xi$  has the representation  $\xi = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_m} - B_{t_{m-1}})$  with  $t = t_i$ , for some  $1 \leq i \leq m$ , and we put*

$$\begin{aligned} \hat{\mathbb{E}}_{t_i}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_m} - B_{t_{m-1}})] \\ = \tilde{\varphi}(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_i} - B_{t_{i-1}}), \end{aligned}$$

*where*

$$\tilde{\varphi}(x_1, \dots, x_i) = \hat{\mathbb{E}}[\varphi(x_1, \dots, x_i, B_{t_{i+1}} - B_{t_i}, \dots, B_{t_m} - B_{t_{m-1}})].$$

Define  $\|\xi\|_{p,G} = (\hat{\mathbb{E}}[|\xi|^p])^{1/p}$  for  $\xi \in \mathcal{H}_T^0$  and  $p \geq 1$ . Then for all  $t \in [0, T]$ ,  $\hat{\mathbb{E}}_t[\cdot]$  is a continuous mapping on  $\mathcal{H}_T^0$  w.r.t. the norm  $\|\cdot\|_{1,G}$ . Therefore it can be extended continuously to the completion  $L_G^1(\Omega_T)$  of  $\mathcal{H}_T^0$  under the norm  $\|\cdot\|_{1,G}$ .

Let  $L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, \dots, B_{t_n}) : n \geq 1, t_1, \dots, t_n \in [0, T], \varphi \in C_{b.Lip}(\mathbb{R}^{d \times n})\}$ , where  $C_{b.Lip}(\mathbb{R}^{d \times n})$  denotes the set of bounded Lipschitz functions on  $\mathbb{R}^{d \times n}$ .

Denis et al. [4] proved that the completions of  $C_b(\Omega_T)$  (the set of bounded continuous function on  $\Omega_T$ ),  $\mathcal{H}_T^0$  and  $L_{ip}(\Omega_T)$  under  $\|\cdot\|_{p,G}$  are the same and we denote them by  $L_G^p(\Omega_T)$ .

For each fixed  $\mathbf{a} \in \mathbb{R}^d$ ,  $B_t^{\mathbf{a}} = \langle \mathbf{a}, B_t \rangle$  is a 1-dimensional  $G_{\mathbf{a}}$ -Brownian motion, where  $G_{\mathbf{a}}(\alpha) = \frac{1}{2}(\sigma_{\mathbf{a}\mathbf{a}^T}^2 \alpha^+ - \sigma_{-\mathbf{a}\mathbf{a}^T}^2 \alpha^-)$ ,  $\sigma_{\mathbf{a}\mathbf{a}^T}^2 = 2G(\mathbf{a}\mathbf{a}^T)$ ,  $\sigma_{-\mathbf{a}\mathbf{a}^T}^2 = -2G(-\mathbf{a}\mathbf{a}^T)$ . Let  $\pi_t^N = \{t_0^N, \dots, t_N^N\}$ ,  $N = 1, 2, \dots$ , be a sequence of partitions of  $[0, t]$  such that  $\mu(\pi_t^N) = \max\{|t_{i+1}^N - t_i^N| : i = 0, \dots, N-1\} \rightarrow 0$ , the quadratic variation process of  $B^{\mathbf{a}}$  is defined by

$$\langle B^{\mathbf{a}} \rangle_t = \lim_{\mu(\pi_t^N) \rightarrow 0} \sum_{j=0}^{N-1} (B_{t_{j+1}^N}^{\mathbf{a}} - B_{t_j^N}^{\mathbf{a}})^2.$$

For each fixed  $\mathbf{a}, \bar{\mathbf{a}} \in \mathbb{R}^d$ , the mutual variation process of  $B^{\mathbf{a}}$  and  $B^{\bar{\mathbf{a}}}$  is defined by

$$\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}} \rangle_t = \frac{1}{4}[\langle B^{\mathbf{a}+\bar{\mathbf{a}}} \rangle_t - \langle B^{\mathbf{a}-\bar{\mathbf{a}}} \rangle_t].$$

**Definition 4.6** Let  $M_G^0(0, T)$  be the collection of processes in the following form: for a given partition  $\{t_0, \dots, t_N\} = \pi_T$  of  $[0, T]$ ,

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j, t_{j+1})}(t),$$

where  $\xi_i \in L_{ip}(\Omega_{t_i})$ ,  $i = 0, 1, 2, \dots, N-1$ . For  $p \geq 1$  and  $\eta \in M_G^0(0, T)$ , let  $\|\eta\|_{H_G^p} = \{\hat{\mathbb{E}}[(\int_0^T |\eta_s|^2 ds)^{p/2}]\}^{1/p}$ ,  $\|\eta\|_{M_G^p} = \{\hat{\mathbb{E}}[\int_0^T |\eta_s|^p ds]\}^{1/p}$  and denote by  $H_G^p(0, T)$ ,  $M_G^p(0, T)$  the completions of  $M_G^0(0, T)$  under the norms  $\|\cdot\|_{H_G^p}$ ,  $\|\cdot\|_{M_G^p}$  respectively.

**Theorem 4.7** ([4, 9]) There exists a weakly compact set  $\mathcal{P} \subset \mathcal{M}_1(\Omega_T)$ , the set of probability measures on  $(\Omega_T, \mathcal{B}(\Omega_T))$ , such that

$$\hat{\mathbb{E}}[\xi] = \sup_{P \in \mathcal{P}} E_P[\xi] \quad \text{for all } \xi \in \mathcal{H}_T^0.$$

$\mathcal{P}$  is called a set that represents  $\hat{\mathbb{E}}$ .

Let  $\mathcal{P}$  be a weakly compact set that represents  $\hat{\mathbb{E}}$ . For this  $\mathcal{P}$ , we define capacity

$$c(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega_T).$$

A set  $A \subset \Omega_T$  is polar if  $c(A) = 0$ . A property holds “quasi-surely” (q.s. for short) if it holds outside a polar set. In the following, we do not distinguish two random variables  $X$  and  $Y$  if  $X = Y$  q.s..

**Definition 4.8** A process  $\{M_t\}$  with values in  $L_G^1(\Omega_T)$  is called a  $G$ -martingale if  $\hat{\mathbb{E}}_s[M_t] = M_s$  for any  $s \leq t$ .

Let  $S_G^0(0, T) = \{h(t, B_{t_1 \wedge t}, \dots, B_{t_n \wedge t}) : t_1, \dots, t_n \in [0, T], h \in C_{b, Lip}(\mathbb{R}^{n+1})\}$ . For  $p \geq 1$  and  $\eta \in S_G^0(0, T)$ , set  $\|\eta\|_{S_G^p} = \{\hat{\mathbb{E}}[\sup_{t \in [0, T]} |\eta_t|^p]\}^{\frac{1}{p}}$ . Denote by  $S_G^p(0, T)$  the completion of  $S_G^0(0, T)$  under the norm  $\|\cdot\|_{S_G^p}$ .

We consider the following type of  $G$ -BSDEs (in this paper we always use Einstein convention):

$$\begin{aligned} Y_t = & \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g_{ij}(s, Y_s, Z_s) d\langle B^i, B^j \rangle_s \\ & - \int_t^T Z_s dB_s - (K_T - K_t), \end{aligned} \quad (4.1)$$

where

$$f(t, \omega, y, z), g_{ij}(t, \omega, y, z) : [0, T] \times \Omega_T \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$$

satisfy the following properties:

**(H1)** There exists some  $\beta > 1$  such that for any  $y, z$ ,  $f(\cdot, \cdot, y, z), g_{ij}(\cdot, \cdot, y, z) \in M_G^\beta(0, T)$ ;

**(H2)** There exists some  $L > 0$  such that

$$|f(t, y, z) - f(t, y', z')| + \sum_{i,j=1}^d |g_{ij}(t, y, z) - g_{ij}(t, y', z')| \leq L(|y - y'| + |z - z'|).$$

For simplicity, we denote by  $\mathfrak{S}_G^\alpha(0, T)$  the collection of processes  $(Y, Z, K)$  such that  $Y \in S_G^\alpha(0, T)$ ,  $Z \in H_G^\alpha(0, T; \mathbb{R}^d)$ ,  $K$  is a decreasing  $G$ -martingale with  $K_0 = 0$  and  $K_T \in L_G^\alpha(\Omega_T)$ .

**Definition 4.9** Let  $\xi \in L_G^\beta(\Omega_T)$  and  $f$  satisfy (H1) and (H2) for some  $\beta > 1$ . A triplet of processes  $(Y, Z, K)$  is called a solution of equation (4.1) if for some  $1 < \alpha \leq \beta$  the following properties hold:

- (a)  $(Y, Z, K) \in \mathfrak{S}_G^\alpha(0, T)$ ;
- (b)  $Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g_{ij}(s, Y_s, Z_s) d\langle B^i, B^j \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t)$ .

**Theorem 4.10** ([7]) Assume that  $\xi \in L_G^\beta(\Omega_T)$  and  $f, g_{ij}$  satisfy (H1) and (H2) for some  $\beta > 1$ . Then equation (4.1) has a unique solution  $(Y, Z, K)$ . Moreover, for any  $1 < \alpha < \beta$  we have  $Y \in S_G^\alpha(0, T)$ ,  $Z \in H_G^\alpha(0, T; \mathbb{R}^d)$  and  $K_T \in L_G^\alpha(\Omega_T)$ .

## 4.1 Comparison theorem of SDEs

Let  $\tau \in [0, T]$  and  $\eta \in L_G^2(\Omega_\tau)$ , we consider the following type SDE:

$$X_t = \eta + \int_\tau^t b(s, X_s)ds + \int_\tau^t h(s, X_s)d\langle B \rangle_s + \int_\tau^t \sigma(s, X_s)dB_s + V_t - V_\tau, \quad (4.2)$$

where  $b, h, \sigma$  are given functions satisfying  $b(\cdot, x), h(\cdot, x), \sigma(\cdot, x) \in M_G^2(\tau, T)$  for each  $x \in \mathbb{R}$  and the Lipschitz condition, i.e.,

$$|b(t, x) - b(t, x')| + |h(t, x) - h(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leq K|x - x'|;$$

$(V_t)_{t \in [\tau, T]}$  is a given RCLL process such that  $\hat{\mathbb{E}}[\sup_{t \in [\tau, T]} |V_t|^2] < \infty$ . Peng [23] proved that the above SDE has a unique solution  $X \in M_G^2(\tau, T)$ .

**Theorem 4.11** *Let  $(X_t^i)_{t \in [\tau, T]}$ ,  $i = 1, 2$ , be the solutions of the following SDEs:*

$$X_t^i = \eta^i + \int_\tau^t b_i(s, X_s^i)ds + \int_\tau^t h_i(s, X_s^i)d\langle B \rangle_s + \int_\tau^t \sigma(s, X_s^i)dB_s + V_t^i - V_\tau^i.$$

*If  $\eta^1 \geq \eta^2$ ,  $b_1 \geq b_2$ ,  $h_1 \geq h_2$ ,  $V_t^1 - V_t^2$  is an increasing process, then  $X_t^1 \geq X_t^2$ .*

**Proof.** We have

$$\hat{X}_t = \hat{\eta} + \int_\tau^t \hat{b}_s ds + \int_\tau^t \hat{h}_s d\langle B \rangle_s + \int_\tau^t \hat{\sigma}_s dB_s + \hat{V}_t - \hat{V}_\tau,$$

where  $\hat{X}_t = X_t^1 - X_t^2$ ,  $\hat{\eta} = \eta^1 - \eta^2$ ,  $\hat{b}_s = b_1(s, X_s^1) - b_2(s, X_s^2)$ ,  $\hat{h}_s = h_1(s, X_s^1) - h_2(s, X_s^2)$ ,  $\hat{\sigma}_s = \sigma(s, X_s^1) - \sigma(s, X_s^2)$ ,  $\hat{V}_t = V_t^1 - V_t^2$ . For each given  $\varepsilon > 0$ , we can choose Lipschitz function  $l(\cdot)$  such that  $I_{[-\varepsilon, \varepsilon]} \leq l(x) \leq I_{[-2\varepsilon, 2\varepsilon]}$ . Thus we have

$$b_1(s, X_s^1) - b_1(s, X_s^2) = (b_1(s, X_s^1) - b_1(s, X_s^2))l(\hat{X}_s) + b_s^\varepsilon \hat{X}_s,$$

where  $b_s^\varepsilon = (1 - l(\hat{X}_s))(b_1(s, X_s^1) - b_1(s, X_s^2))\hat{X}_s^{-1} \in M_G^2(\tau, T)$  such that  $|b_s^\varepsilon| \leq K$ . It is easy to verify that

$$|(b_1(s, X_s^1) - b_1(s, X_s^2))l(\hat{X}_s)| \leq K|\hat{X}_s|l(\hat{X}_s) \leq 2K\varepsilon.$$

Thus we can get

$$\hat{b}_s = b_s^\varepsilon \hat{X}_s + m_s + m_s^\varepsilon, \quad \hat{h}_s = h_s^\varepsilon \hat{X}_s + n_s + n_s^\varepsilon, \quad \hat{\sigma}_s = \sigma_s^\varepsilon \hat{X}_s + l_s^\varepsilon,$$

where  $|m_s^\varepsilon| \leq 2K\varepsilon$ ,  $|n_s^\varepsilon| \leq 2K\varepsilon$ ,  $|l_s^\varepsilon| \leq 2K\varepsilon$ ,  $m_s = b_1(s, X_s^1) - b_2(s, X_s^2) \geq 0$  and  $n_s = h_1(s, X_s^1) - h_2(s, X_s^2) \geq 0$ . Let  $\Gamma_t^\varepsilon$  be the solution of the following SDE:

$$\Gamma_t^\varepsilon = 1 - \int_\tau^t b_s^\varepsilon \Gamma_s^\varepsilon ds - \int_\tau^t [h_s^\varepsilon - (\sigma_s^\varepsilon)^2] \Gamma_s^\varepsilon d\langle B \rangle_s - \int_\tau^t \sigma_s^\varepsilon \Gamma_s^\varepsilon dB_s.$$

By applying Itô's formula to  $\Gamma_t^\varepsilon \hat{X}_t$ , we obtain that

$$\hat{X}_t \geq (\Gamma_t^\varepsilon)^{-1} \left[ \int_\tau^t m_s^\varepsilon \Gamma_s^\varepsilon ds + \int_\tau^t (n_s^\varepsilon - \sigma_s^\varepsilon l_s^\varepsilon) \Gamma_s^\varepsilon d\langle B \rangle_s + \int_\tau^t l_s^\varepsilon \Gamma_s^\varepsilon dB_s \right].$$

Note that  $\Gamma_t^\varepsilon = \exp(-\int_\tau^t b_s^\varepsilon ds - \int_\tau^t [h_s^\varepsilon - \frac{1}{2}(\sigma_s^\varepsilon)^2] d\langle B \rangle_s - \int_\tau^t \sigma_s^\varepsilon dB_s)$ , thus we can get  $\hat{X}_t \geq 0$  by letting  $\varepsilon \rightarrow 0$ .  $\square$

## 4.2 Girsanov transformation

We consider the following  $G$ -BSDE:

$$Y_t = \xi + \int_t^T b_s Z_s ds + \int_t^T d_s Z_s d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t),$$

where  $(b_t)_{t \leq T}$  and  $(d_t)_{t \leq T}$  are bounded processes. For each  $\xi \in L_G^\beta(\Omega_T)$  with  $\beta > 1$ , define

$$\tilde{\mathbb{E}}_t[\xi] = Y_t.$$

By Theorem 5.1 in [8], we know that  $\tilde{\mathbb{E}}_t[\cdot]$  is a consistent sublinear expectation.

**Theorem 4.12** ([8]) *Let  $(b_t)_{t \leq T}$  and  $(d_t)_{t \leq T}$  be bounded processes. Then*

- (1)  $\tilde{B}_t := B_t - \int_0^t b_s ds - \int_0^t d_s d\langle B \rangle_s$  is a  $G$ -Brownian motion under  $\tilde{\mathbb{E}}$ ;
- (2) for any decreasing  $G$ -martingale  $\tilde{K}$  with  $\tilde{K}_0 = 0$  and  $\tilde{K}_T \in L_G^\beta(\Omega_T)$  for some  $\beta > 1$ , we have  $\tilde{\mathbb{E}}_t[\tilde{K}_T] = \tilde{K}_t$ ;
- (3) the quadratic variation process of  $\tilde{B}$  under  $\tilde{\mathbb{E}}$  equals to  $\langle B \rangle$ .

**Proof.** (1) and (2) can be found in [8]. We only prove (3). For each fixed  $t > 0$ , it is easy to check that

$$\lim_{n \rightarrow \infty} \hat{\mathbb{E}}\left[\left|\sum_{i=0}^{n-1} |\tilde{B}_{\frac{i+1}{n}t} - \tilde{B}_{\frac{i}{n}t}|^2 - \langle B \rangle_t\right|^2\right] = 0.$$

By Proposition 3.7 in [7], we can get  $\tilde{\mathbb{E}}\left[\left|\sum_{i=0}^{n-1} |\tilde{B}_{\frac{i+1}{n}t} - \tilde{B}_{\frac{i}{n}t}|^2 - \langle B \rangle_t\right|^\beta\right] \rightarrow 0$  as  $n \rightarrow \infty$  for some  $\beta \in (1, 2)$ . On the other hand,  $\tilde{\mathbb{E}}\left[\left|\sum_{i=0}^{n-1} |\tilde{B}_{\frac{i+1}{n}t} - \tilde{B}_{\frac{i}{n}t}|^2 - \langle \tilde{B} \rangle_t\right|^\beta\right] \rightarrow 0$  as  $n \rightarrow \infty$ . Thus  $\langle \tilde{B} \rangle_t = \langle B \rangle_t$  under  $\tilde{\mathbb{E}}$ .  $\square$

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